Functions

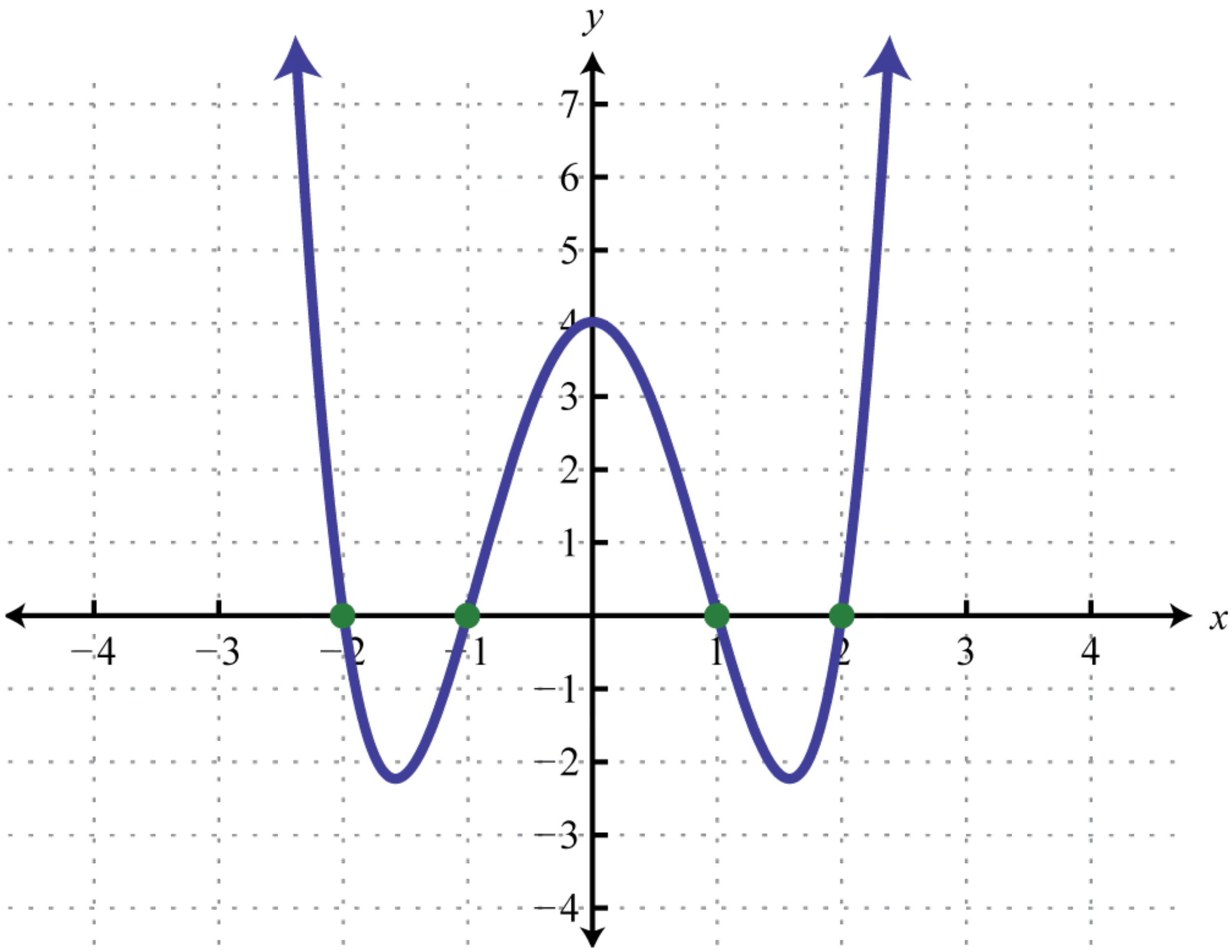
Part One

Outline for Today

- ***What is a Function?***
  - It's more nuanced than you might expect.
- ***Domains and Codomains***
  - Where functions start, and where functions end.
- ***Defining a Function***
  - Expressing transformations compactly.
- ***Special Classes of Functions***
  - Useful types of functions you'll encounter IRL.
- ***Proofs on First-Order Definitions***
  - A key skill.

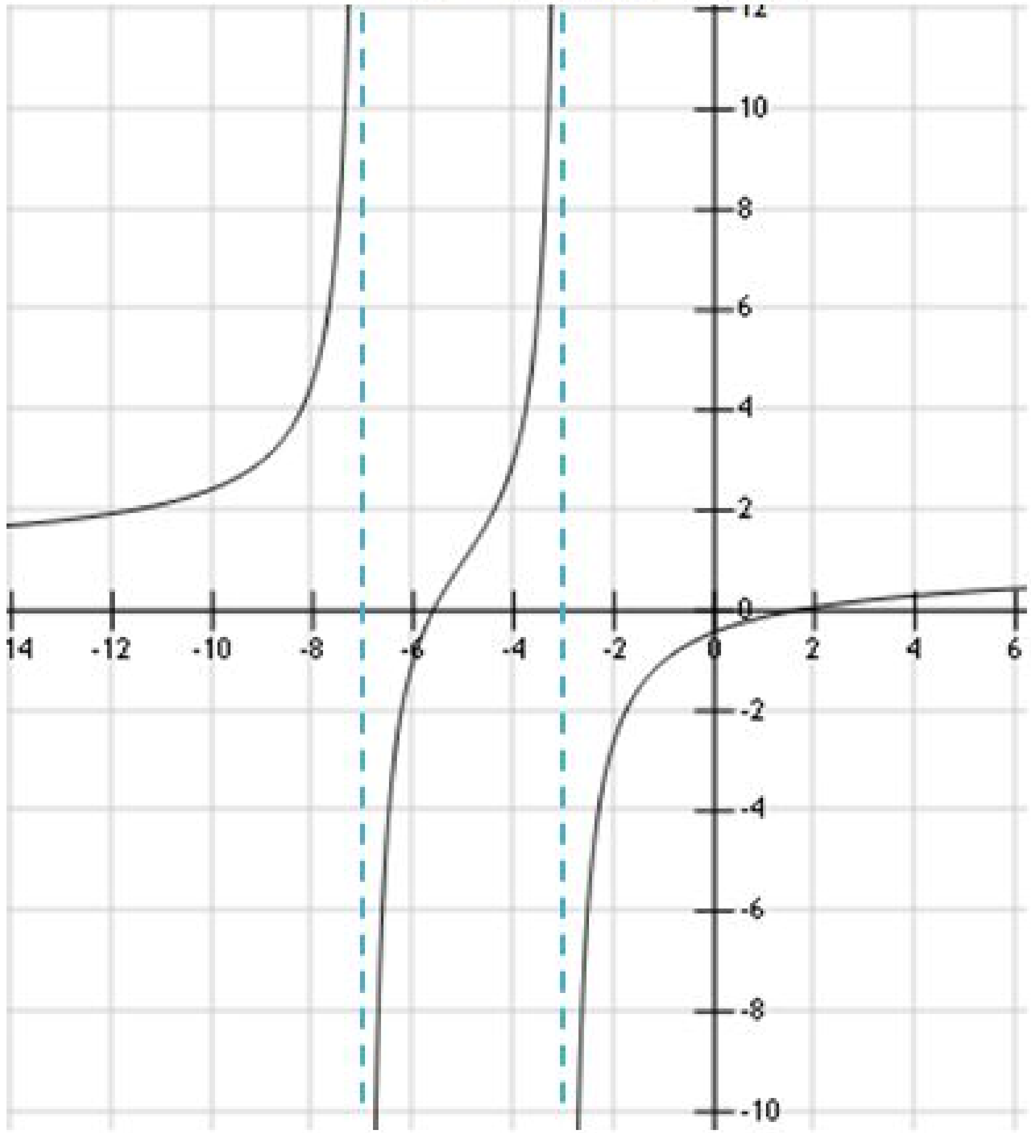
What is a function?

Functions, High-School Edition



$$f(x) = x^4 - 5x^2 + 4$$

$$f(x) = \frac{x^2 + 4x - 9}{x^2 + 10x + 21}$$



source: http://study.com/cimages/multimages/16/asymptote_1.JPG

Functions, High-School Edition

- In high school, functions are usually given as objects of the form

$$f(x) = \frac{x^3 + 3x^2 + 15x + 7}{1 - x^{137}}$$

- What does a function do?
  - It takes in as input a real number.
  - It outputs a real number
  - … except when there are vertical asymptotes or other discontinuities, in which case the function doesn't output anything.

Functions, CS Edition

```
int flipUntil(int n) {
  int numHeads = 0;
  int numTries = 0;

  while (numHeads < n) {
    if (randomBoolean()) {
      numHeads++;
    }
    numTries++;
  }

  return numTries;
}
```

Functions, CS Edition

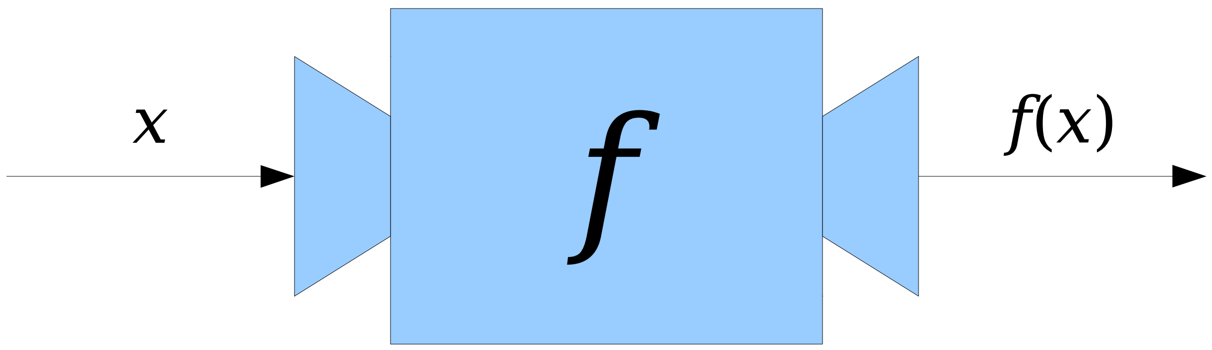
- In programming, functions
  - might take in inputs,
  - might return values,
  - might have side effects,
  - might never return anything,
  - might crash, and
  - might return different values when called multiple times.

What's Common?

- Although high-school math functions and CS functions are pretty different, they have two key aspects in common:
  - They take in inputs.
  - They produce outputs.
- In math, we like to keep things easy, so that's pretty much how we're going to define a function.

High-Level Intuition:

A function is an object f that takes in exactly one input x and produces exactly one output $f(x)$.



(This is not definition. It's just to help you build and intuition.)

In mathematics, functions are ***deterministic***.

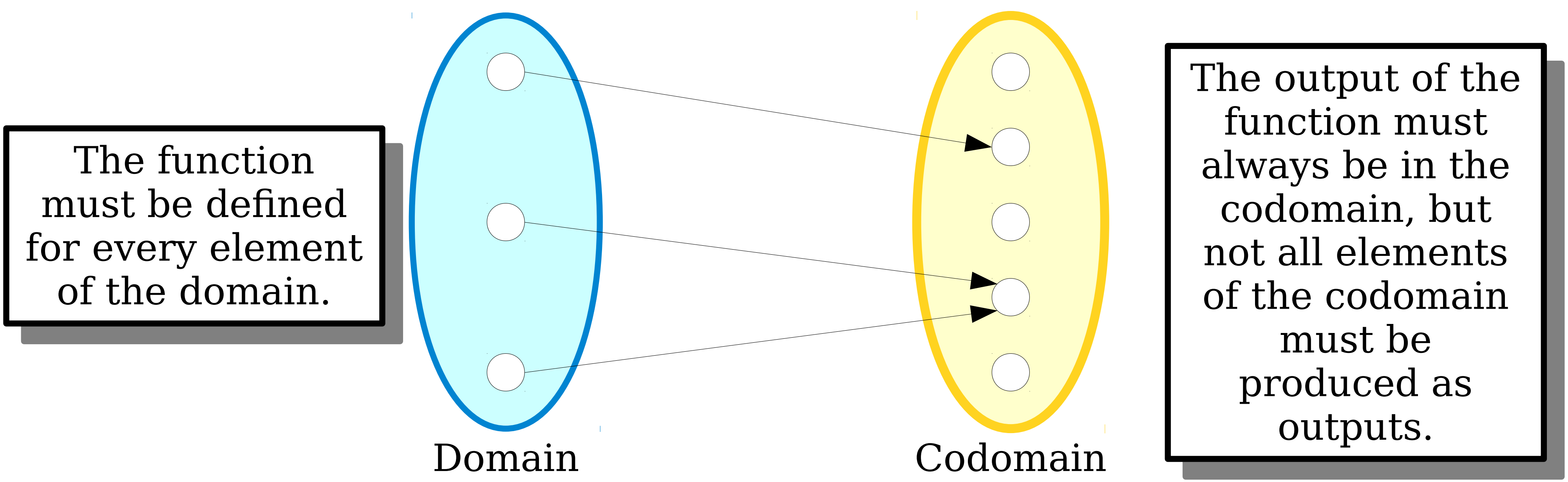
That is, given the same input, a function must always produce the same output.

The following is a perfectly valid piece of C++ code, but it's not a valid function under our definition:

```
int randomNumber(int numOutcomes) {
    return rand() % numOutcomes;
}
```

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.



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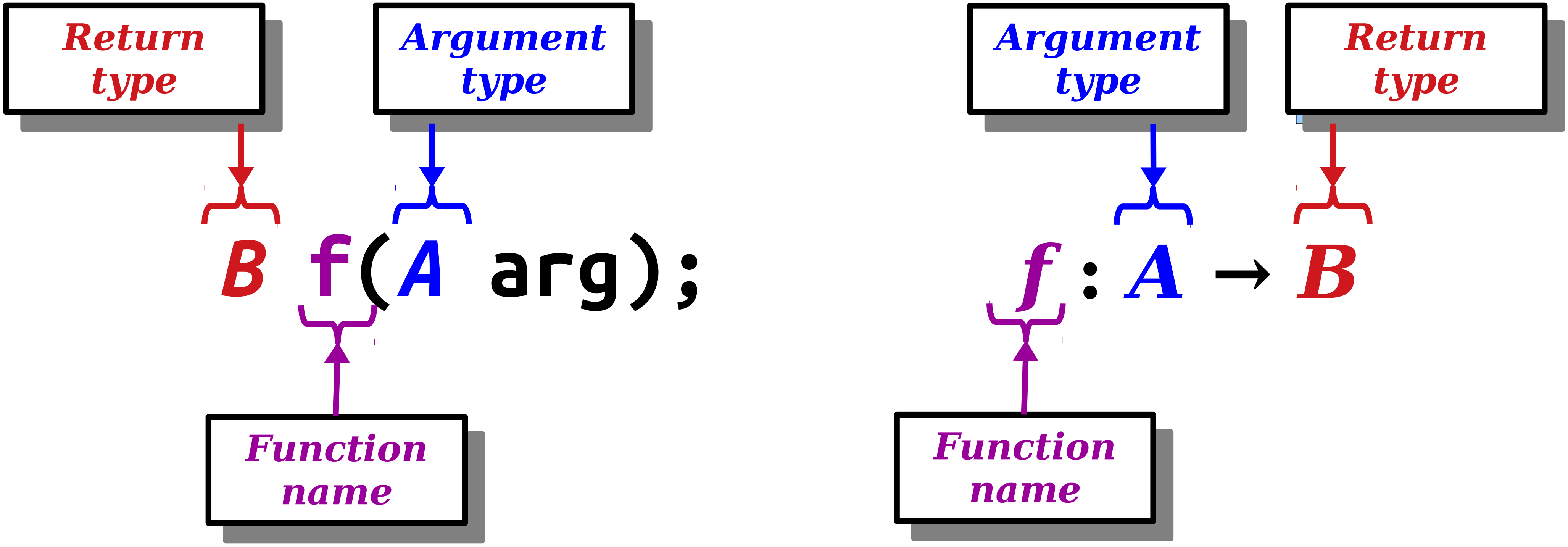
The ***domain*** of this function is ℝ. Any real number can be provided as input.

The ***codomain*** of this function is ℝ. Everything produced is a real number, but not all real numbers can be produced.

```
double absoluteValueOf(double x) {
    if (x >= 0) {
        return x;
    } else {
        return -x;
    }
}
```

Domains and Codomains

- If f is a function whose domain is A and whose codomain is B, we write $f : A \to B$.
- Think of this like a “function prototype” in C++.



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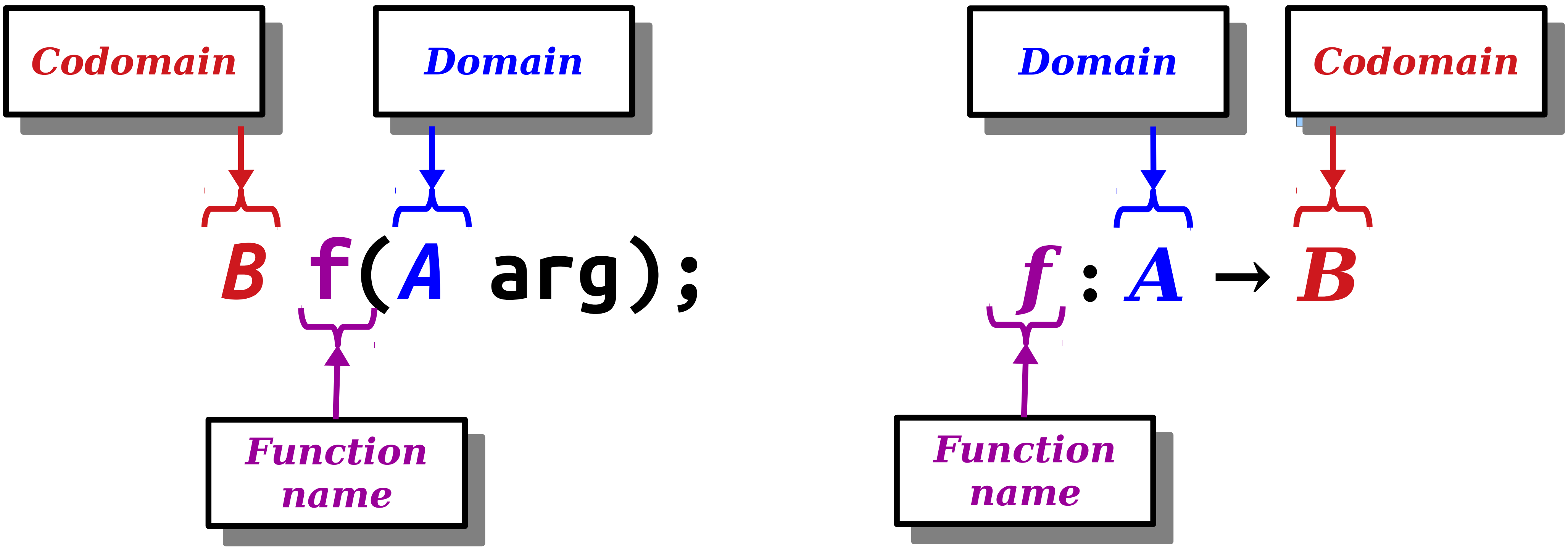


Some Observations

- Usually, when working with functions, you pick the domain and codomain before defining the rule for the function.
  - Think programming: you usually know what types of things you're working with before you know how they work.
- In mathematics, all functions take in exactly one argument: an element of the domain.
- In mathematics, functions are ***deterministic*** and can't behave randomly.

The Official Rules for Functions

- Formally speaking, we say that $f : A \to B$ if the following two rules hold.
- First, f must be obey its domain/codomain rules:

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

(*"Every input in A maps to some output in B."*)

- Second, f must be deterministic:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

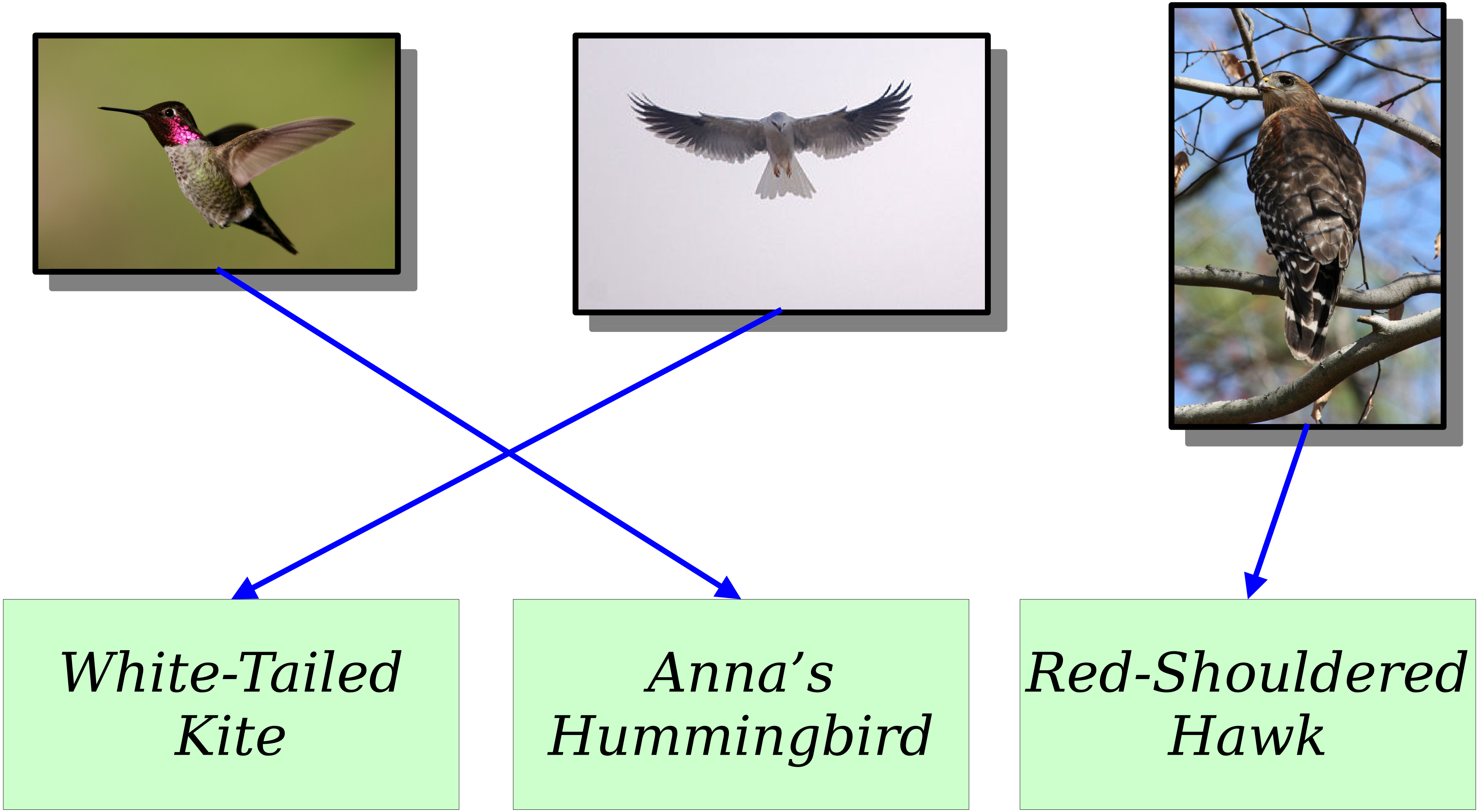
(*"Equal inputs produce equal outputs."*)

- If you're ever curious about whether something is a function, look back at these rules and check! For example:
  - Can a function have an empty domain?
  - Can a function have an empty codomain?

Defining Functions

Defining Functions

- To define a function, you need to
  - specify the domain,
  - specify the codomain, and
  - give a ***rule*** used to evaluate the function.
- All three pieces are necessary.
  - We need to domain to know what the function can be applied to.
  - We need to codomain to know what the output space is.
  - We need the rule to be able to evaluate the function.
- There are many ways to do this. Let’s go over a few examples.



Functions can be defined as a ***picture***.
Draw the domain and codomain explicitly.
Then, add arrows to show the outputs.

$$f : \mathbb{Z} \to \mathbb{Z}, \text{ where}$$

$$f(x) = x^2 + 3x - 15$$

Functions can be defined as a ***rule***.
Be sure to explicitly state what the domain and codomain are!

$$f : \mathbb{Z} \to \mathbb{N}, \text{ where}$$

$$f(n) = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n \leq 0 \end{cases}$$

Some rules are given ***piecewise***. We select which rule to apply based on the conditions on the right. (Just make sure at least one condition applies and that all applicable conditions give the same result!)

Some Nuances

Defining Functions

- Which of the following are functions?
  - $f : \mathbb{R} \to \mathbb{R}$ defined as : $f(x) = \dfrac{x+2}{x+1}$
  - $f : \mathbb{N} \to \mathbb{R}$ defined as : $f(x) = \dfrac{x+2}{x+1}$

Answer at

https://pollev.com/cs103

Rules for defining functions:

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

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This expression isn't defined when x = -1, so f isn't defined over its full domain. We therefore don't consider it to be a function.

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This is a function! Every natural number maps to some real number.

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Time-Out for Announcements!

Problem Set One Graded

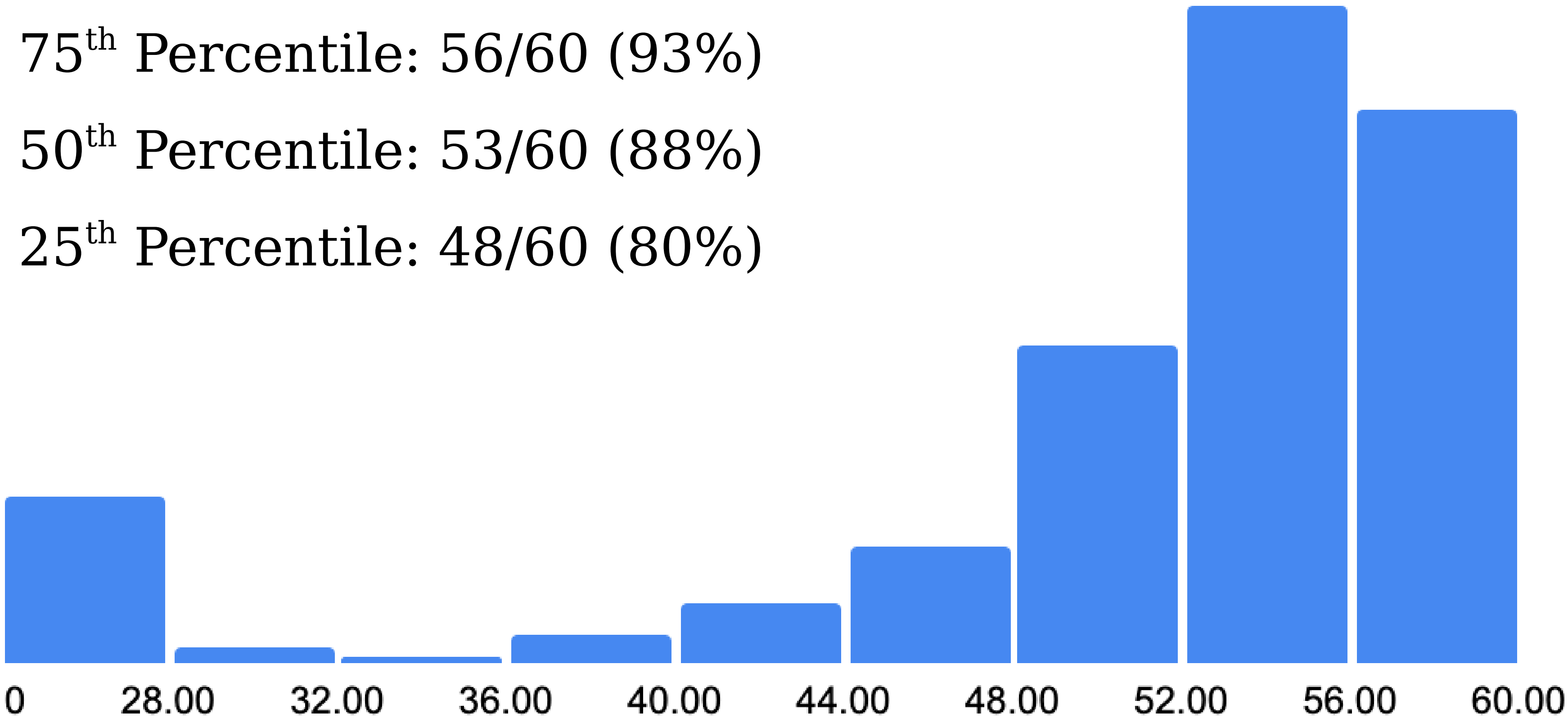
- Your wonderful TAs have finished grading Problem Set One.
- Grades and feedback are up on the Gradescope.
- Solutions are available online on the course website (visit the page for PS1 to get the link).

Problem Set 1 Graded

75th Percentile: 56/60 (93%)

50th Percentile: 53/60 (88%)

25th Percentile: 48/60 (80%)



Pro tips when reading a grading distribution:

1. Standard deviations are ***malicious lies***. Ignore them.
2. The average score is a ***malicious lie***. Ignore it.
3. Raw scores are ***malicious lies***. Ignore them.

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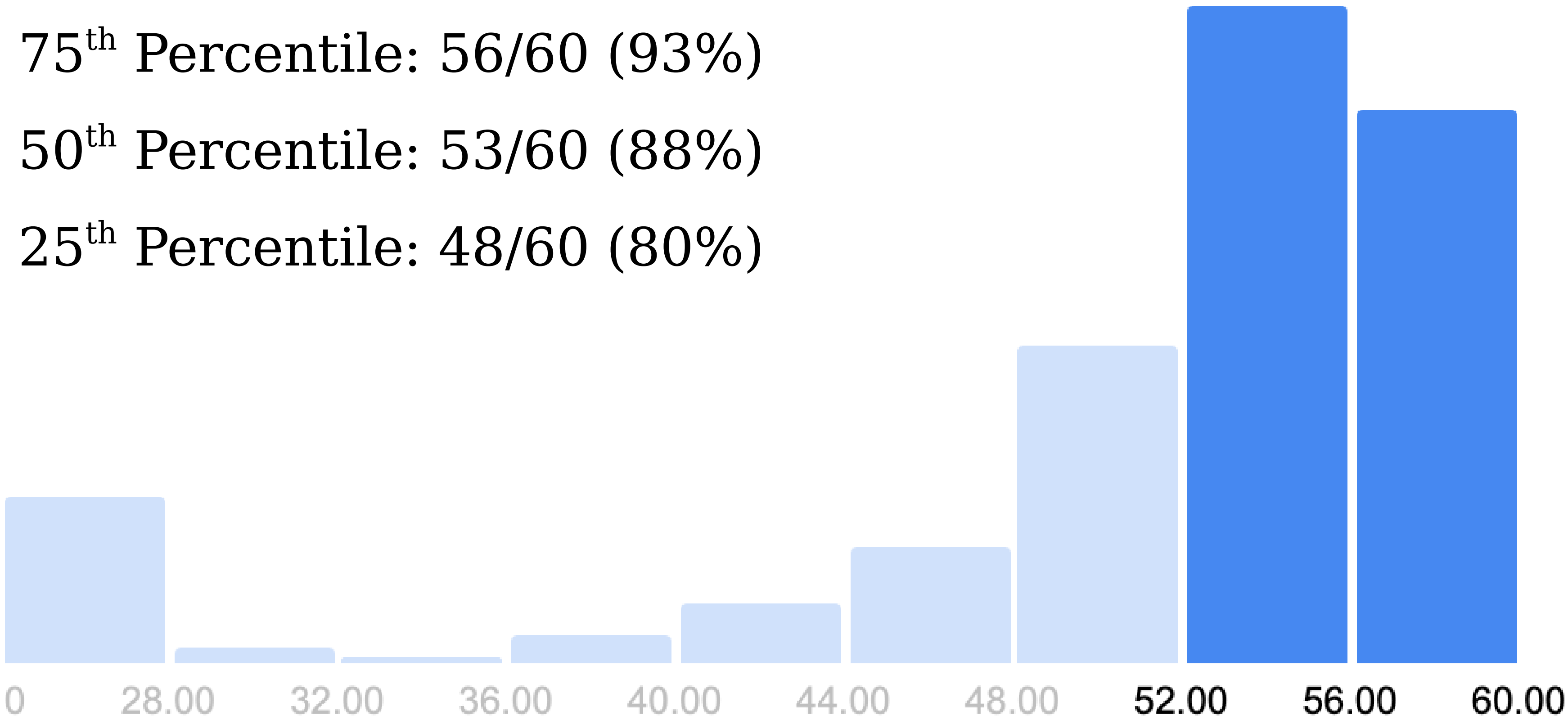

"Great job! Look over your feedback for some tips on how to tweak things for next time."

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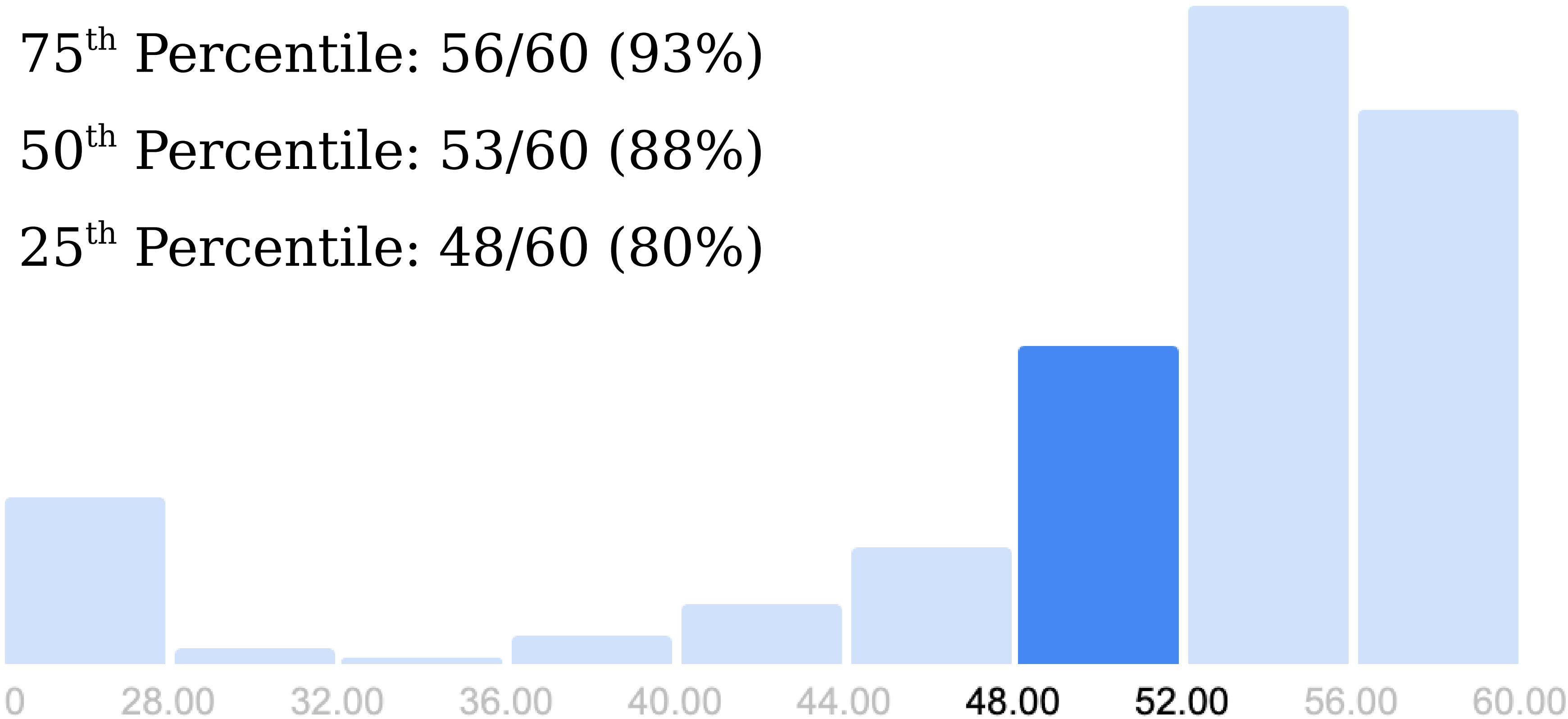

"You're almost there! Review the feedback on your submission and see what to focus on for next time."

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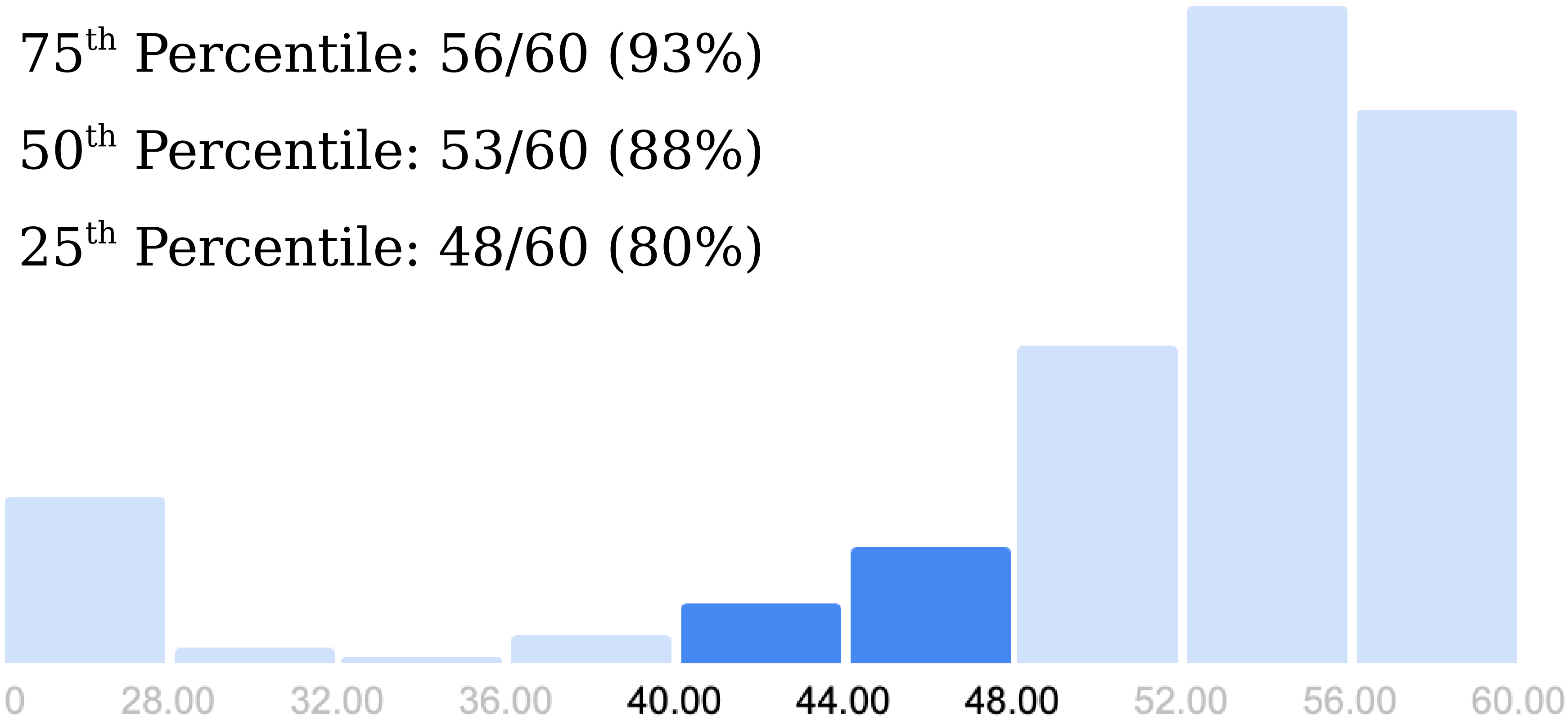

"You're on the right track, but there are some areas where you need to improve. Review your feedback and ask us questions when you have them."

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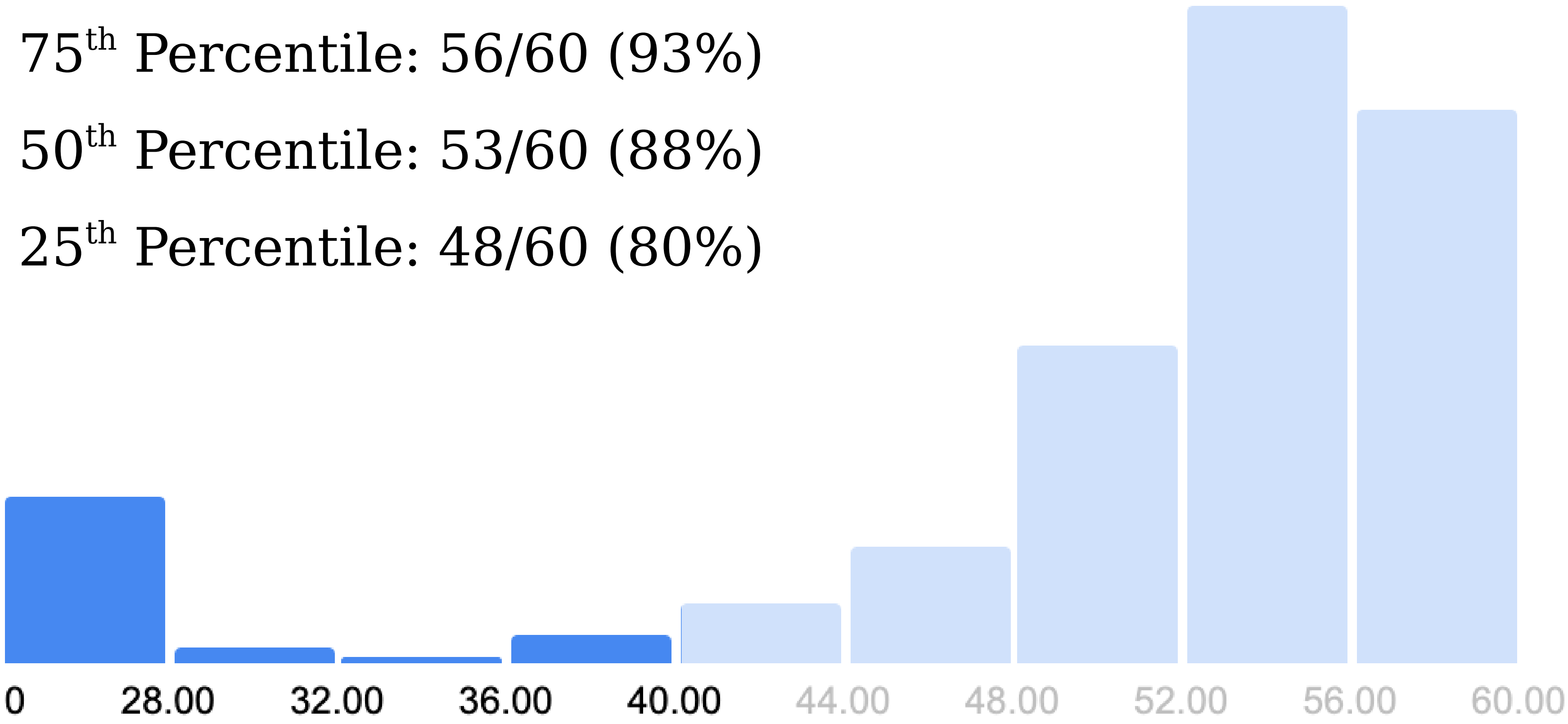
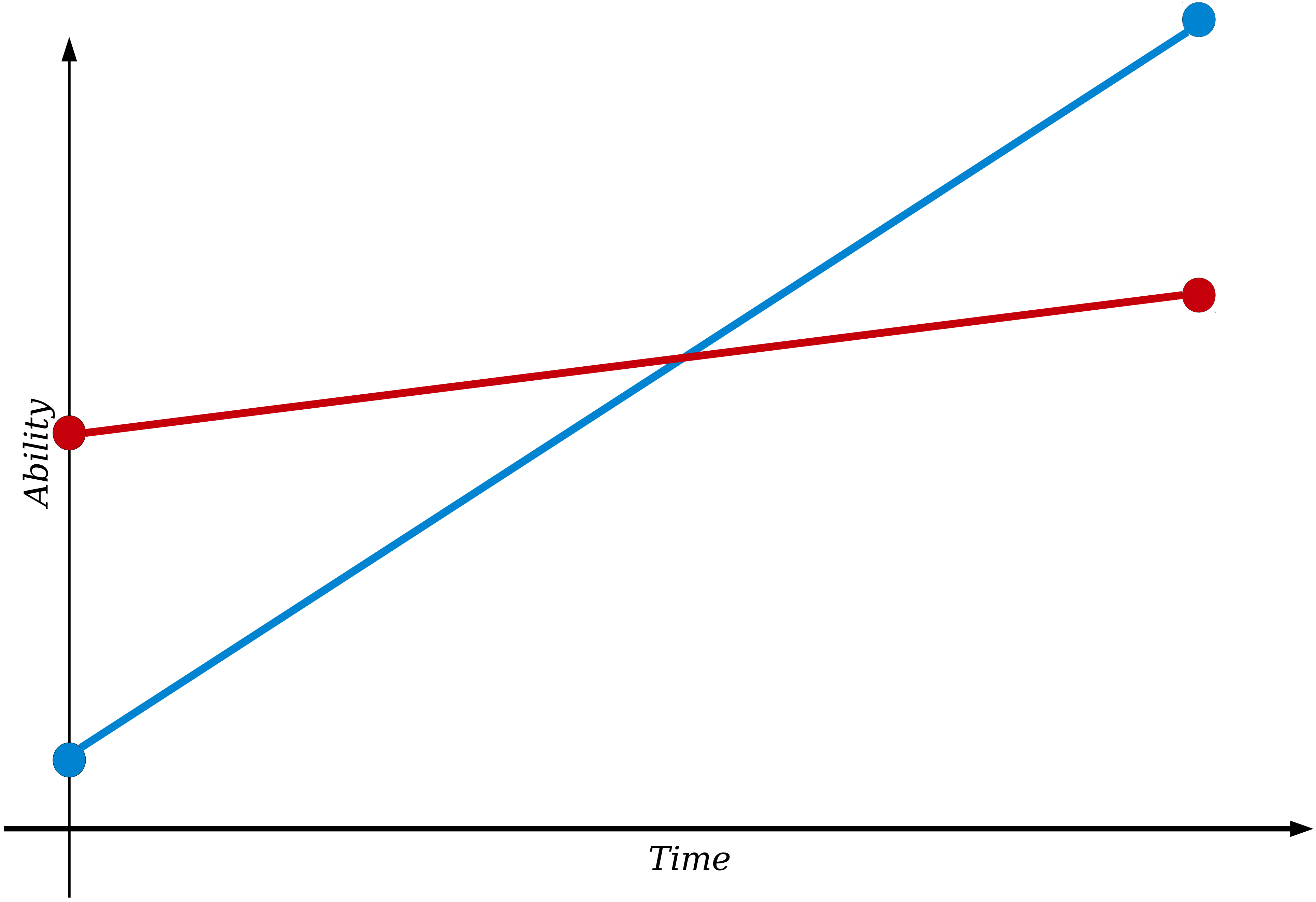


"Looks like something hasn't quite clicked yet. Get in touch with us and stop by office hours to get some extra feedback and advice. Don't get discouraged – you can do this!"



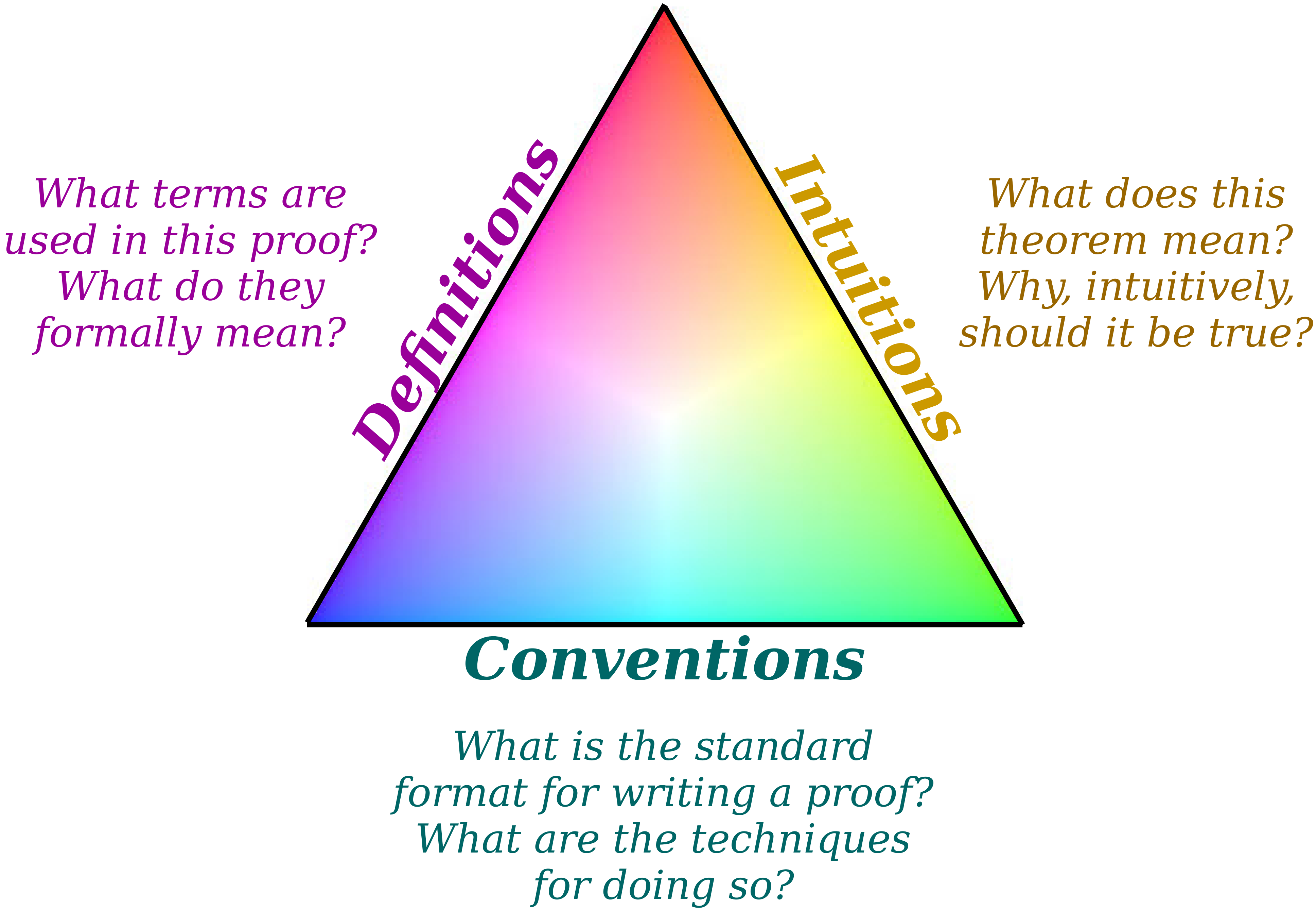
“A little slope makes up for a lot of *y*-intercept.”
- John Ousterhout

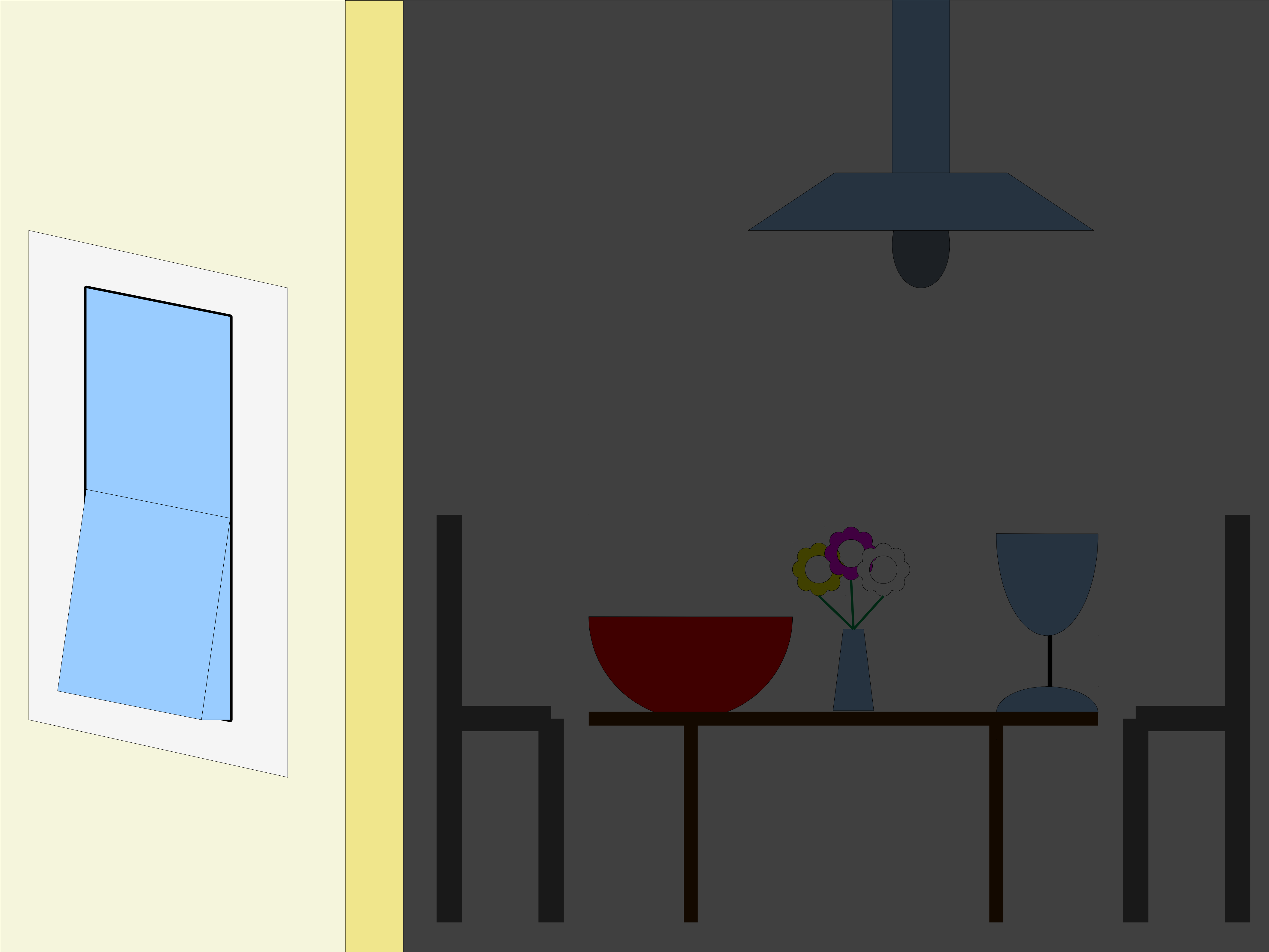
Regrade Requests

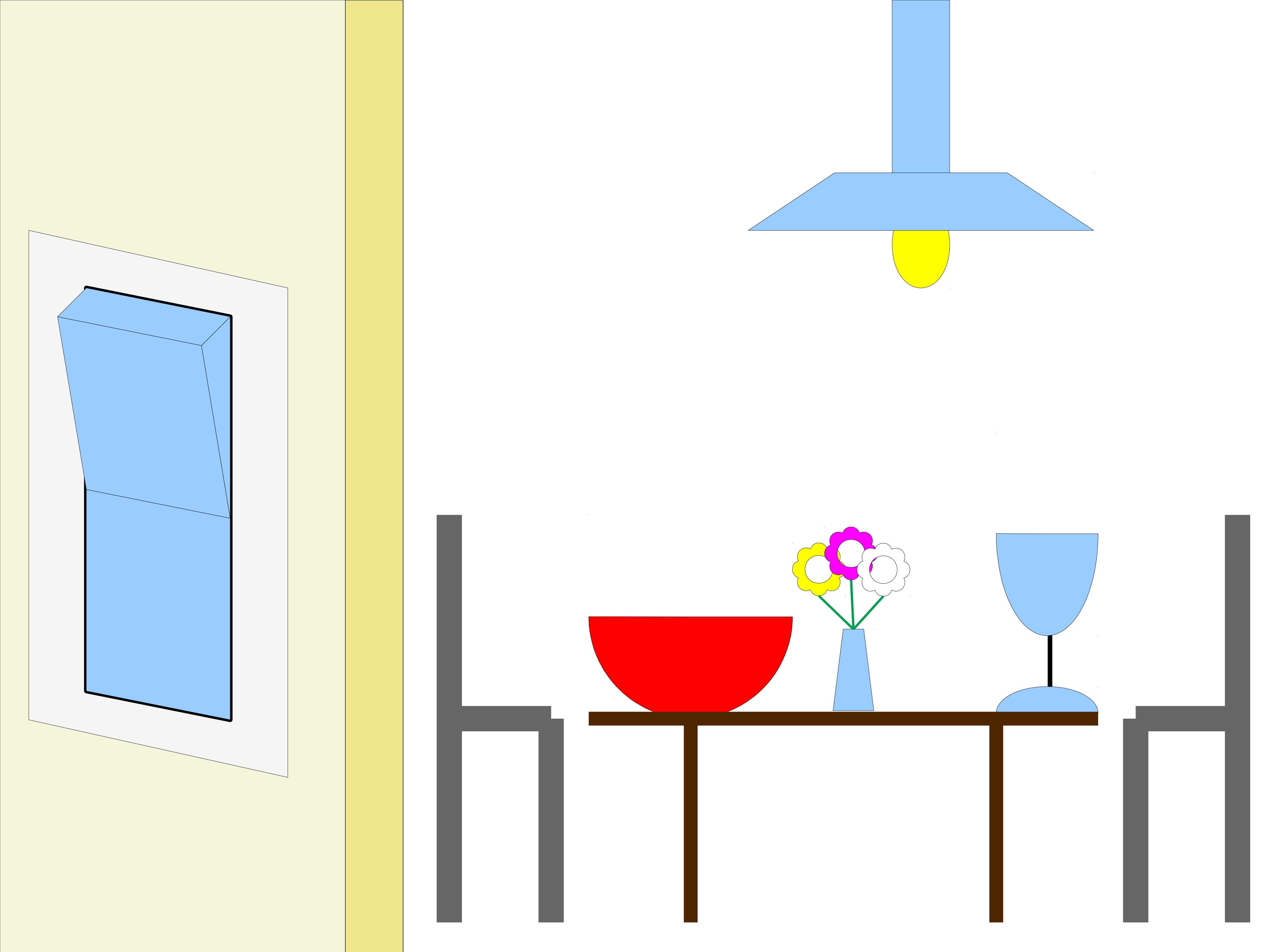
- We’re human. We make mistakes. And we’re happy to correct them!
- Regrades close one week after grades are released.
- Notes on regrades:
  - Please be civil. We make mistakes. We’re happy to correct them.
  - We have to grade what you submitted; we can’t take any clarifications into account during regrades.
  - Regrades are for where we made deductions we shouldn’t have, rather than for the magnitude of deductions.

Back to CS103!

Special Types of Functions


Definitions
Intuitions
Conventions
What terms are used in this proof? What do they formally mean?
What does this theorem mean? Why, intuitively, should it be true?
What is the standard format for writing a proof? What are the techniques for doing so?





Undoing by Doing Again

- Some operations invert themselves. For example:
  - Flipping a switch twice is the same as not flipping it at all.
  - In first-order logic, $\neg\neg A$ is equivalent to A.
  - In algebra, $-(-x) = x$.
  - In set theory, $(A \Delta B) \Delta B = A$. *(Yes, really!)*
- Operations with these properties are surprisingly useful in CS theory and come up in a bunch of contexts.
  - Storing compressed approximations of sets (XOR filters).
  - Building encryption systems (symmetric block ciphers).
  - Transmitting a large file to multiple receivers (fountain codes).

Involutions

- A function $f : A \to A$ from a set back to itself is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")

- Involutions have lots of interesting properties. Let's explore them and see what we can find.

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$.
  - $g : \mathbb{Z} \to \mathbb{Z}$ defined as $g(x) = -x$.
  - $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) = {}^1/_x$.
  - $p : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$p(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

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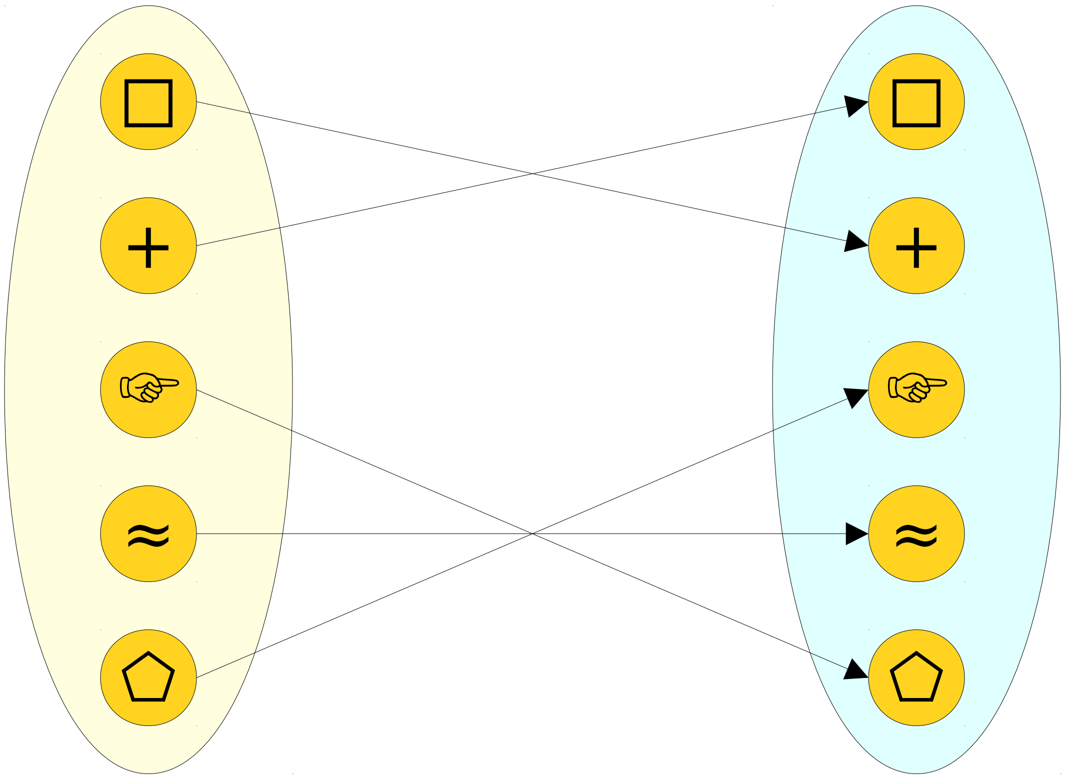
Involutions, Visually



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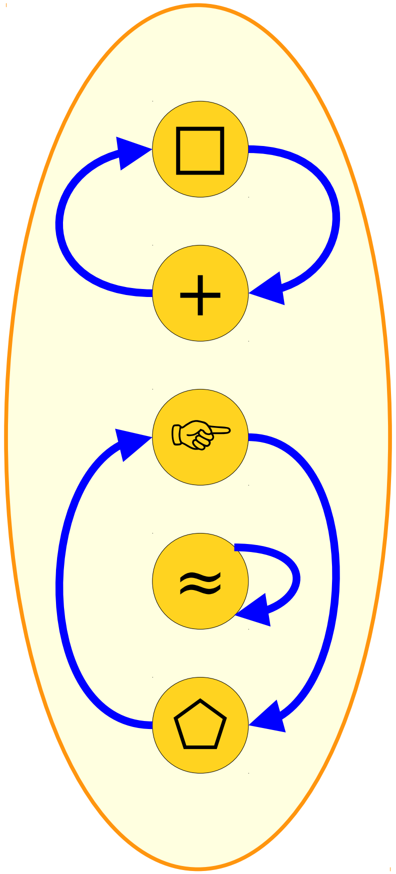

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Proofs on Involutions

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

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Proof: Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$.

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Proof: Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$. To do so, we consider two cases.

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$$\forall n \in \mathbb{N}.\ f(f(n)) = n.$$

What is the negation of this statement?

$$\neg \forall n \in \mathbb{N}.\ f(f(n)) = n$$
$$\exists n \in \mathbb{N}.\ \neg(f(f(n)) = n)$$
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Proof: We need to show that there is some $n \in \mathbb{N}$ where $f(f(n)) \neq n$.

Pick $n = 2$. Then

$$\begin{aligned} f\,(f(n)) &= f(f(2)) \\ &= f(4) \\ &= 16, \end{aligned}$$

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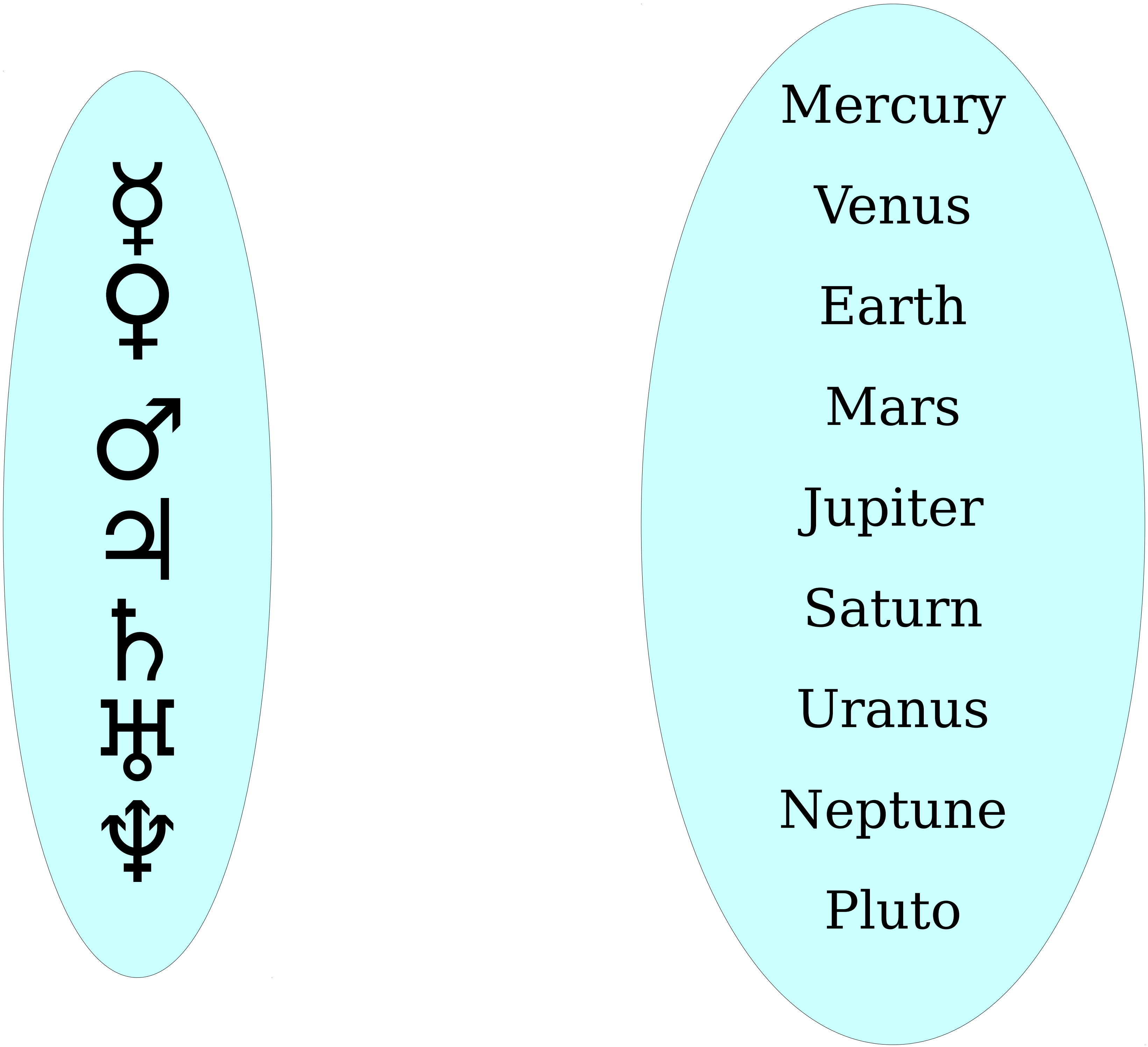
Another Class of Functions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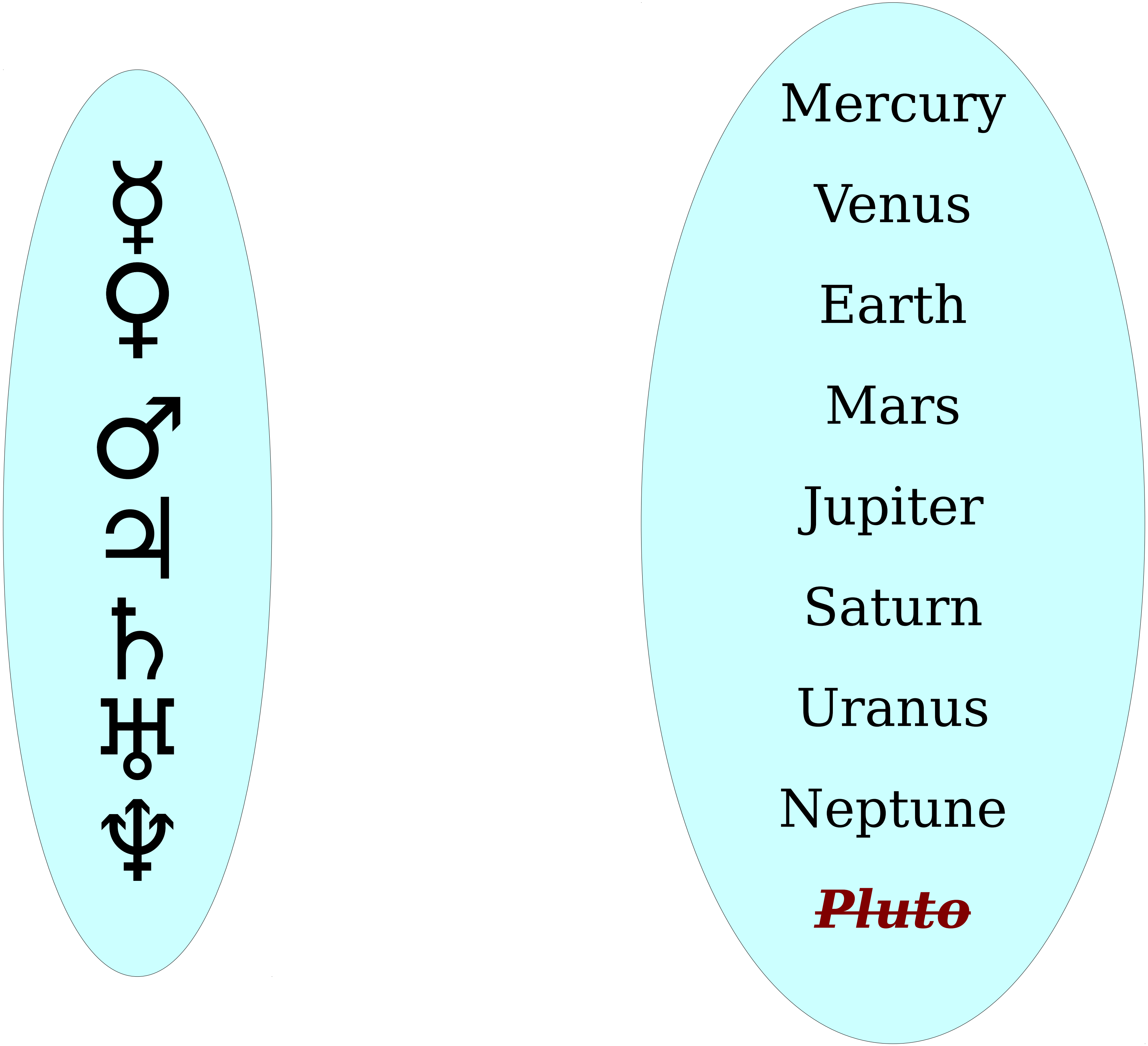
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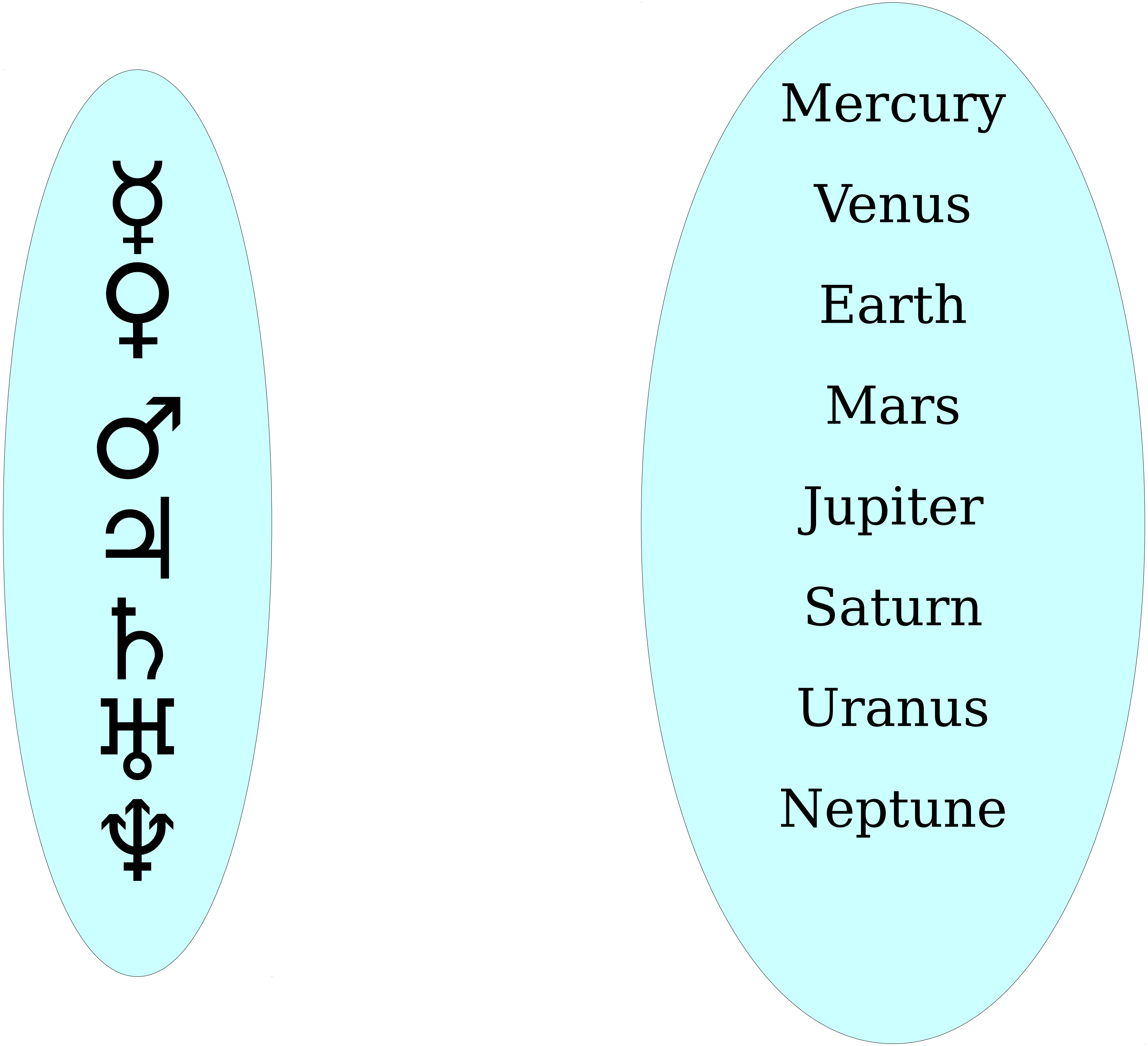
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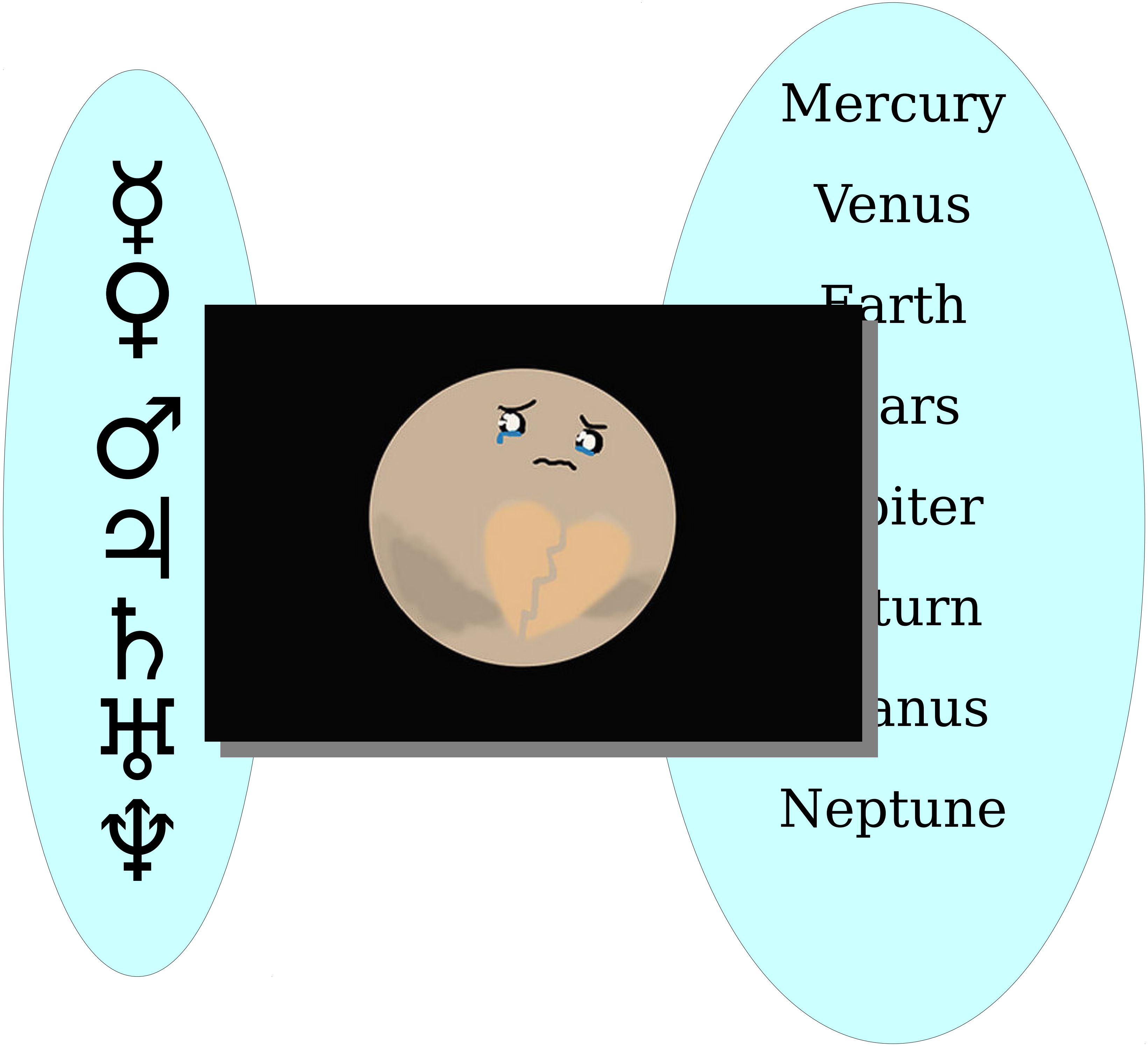
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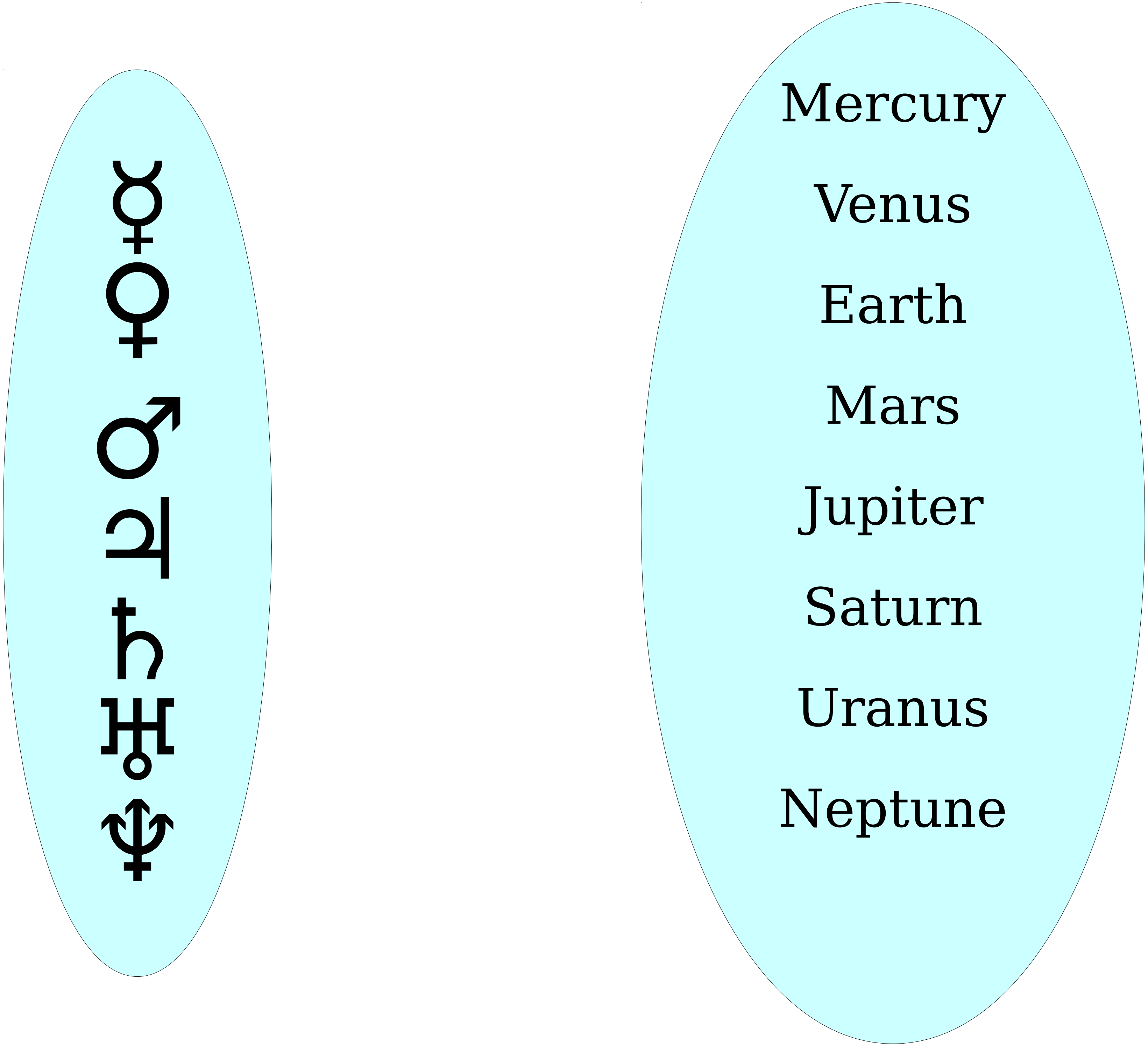
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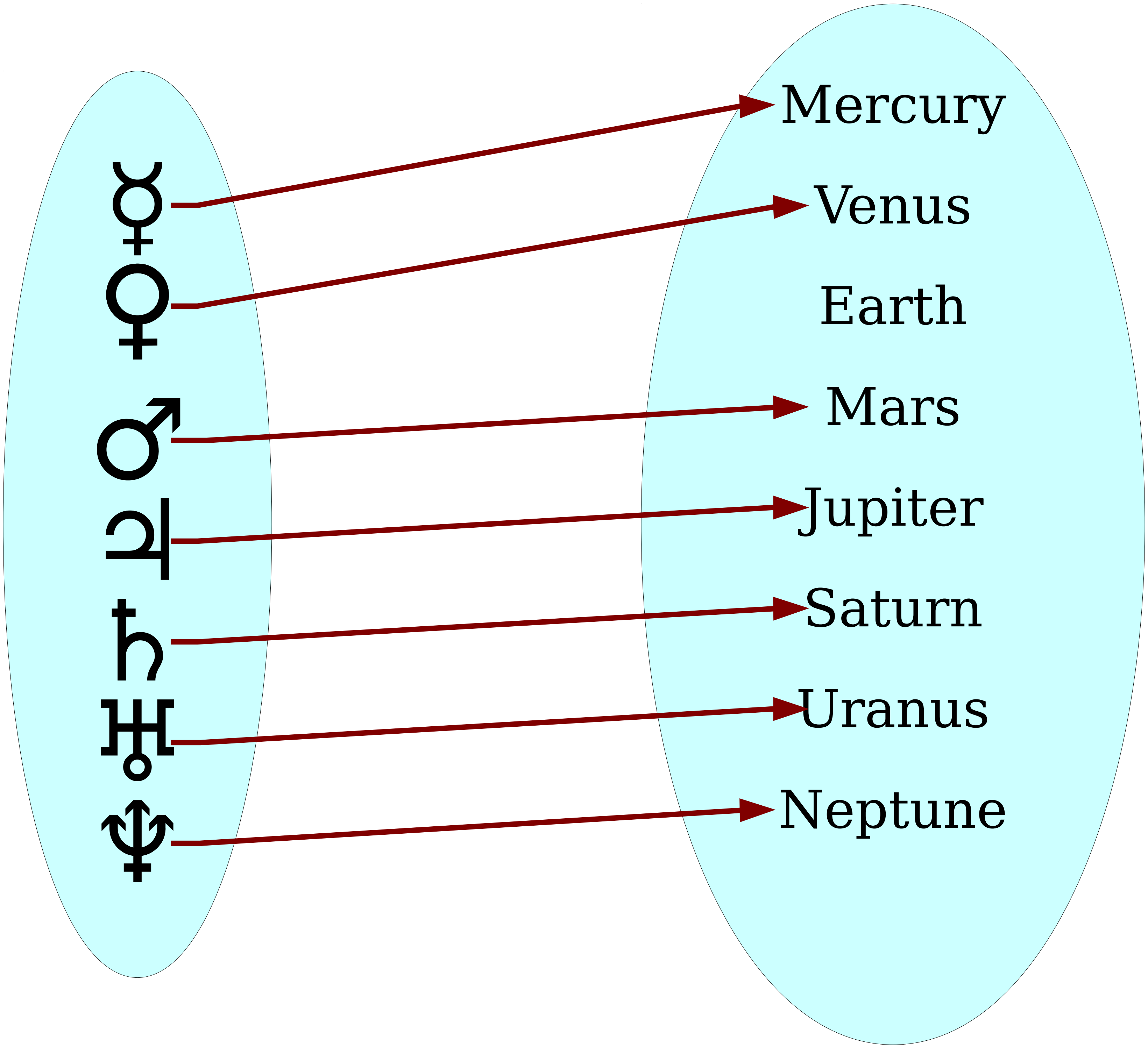
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Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if the following statement is true about f:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different."*)

- The following first-order definition is equivalent *(why?)* and is often useful in proofs.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same."*)

- A function with this property is called an ***injection***.
- How does this compare to our second rule for functions?

Proofs on Injections

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

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Good exercise: Repeat this proof using the other definition of injectivity!

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$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	
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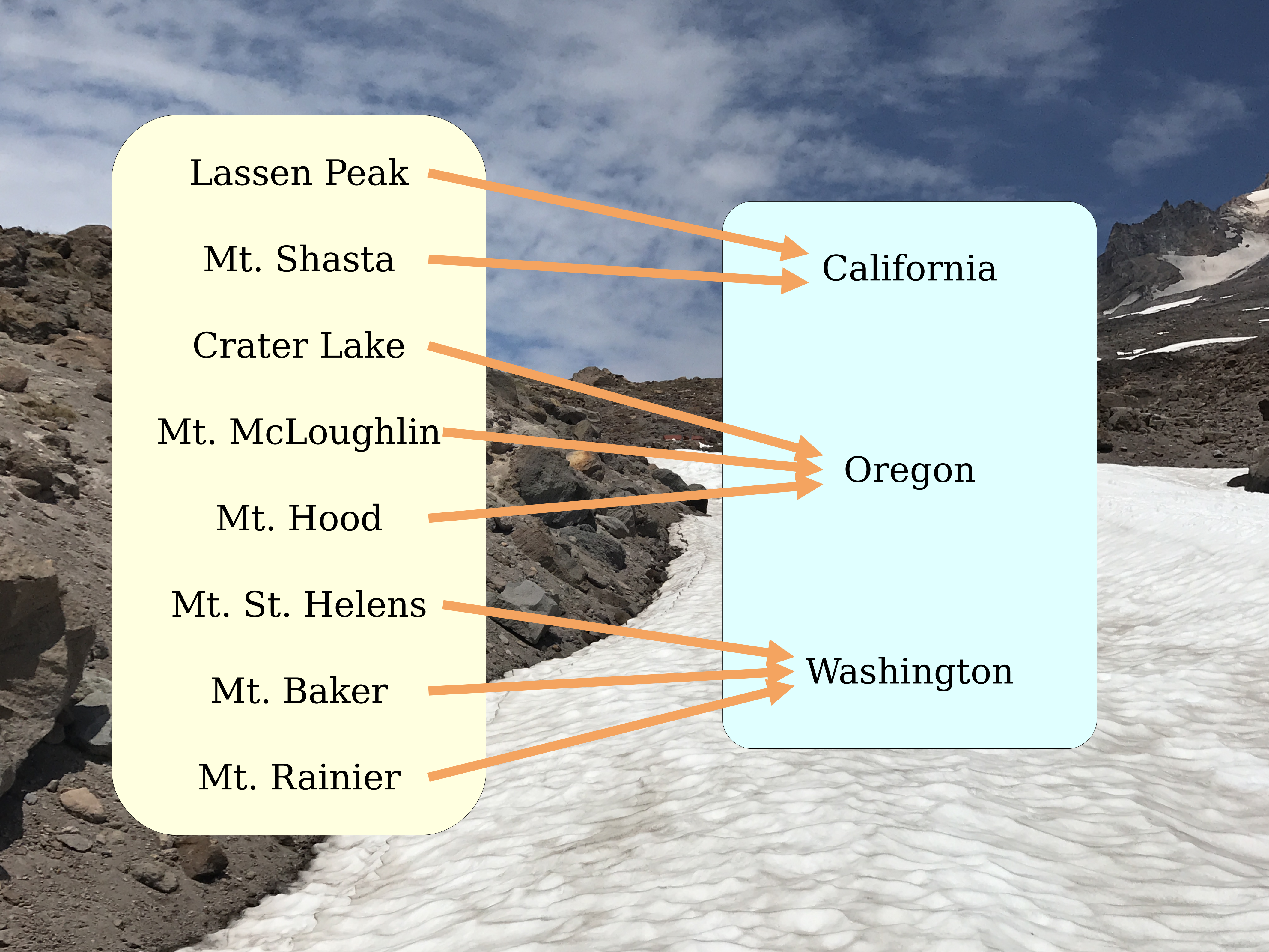
$A \to B$	Assume A is true, then prove B is true.	
$\neg A$	Simplify the negation, then consult this table on the result.	

Another Class of Functions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Mt. Shasta
Crater Lake
Mt. McLoughlin
Mt. Hood
Mt. St. Helens
Mt. Baker
Mt. Rainier
California
Oregon
Washington

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if this first-order logic statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every output, there's an input that produces it."*)

- A function with this property is called a ***surjection***.
- How does this compare to our first rule of functions?

Check the appendix for sample proofs involving injections.

Next Time

- ***First-Order Assumptions***
  - The difference between assuming something is true and proving something is true.
- ***Connecting Function Types***
  - Involutions, injections, and surjections are related to one another. How?
- ***Function Composition***
  - Sequencing functions together.

Appendix: More Proofs on Functions

Proof 1: Proving a function is not injective.

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

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What is the negation of this statement?

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$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \land f(x_1) = f(x_2))$$

Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$. Can we do that?

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What is the negation of this statement?

$$\neg\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
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Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$. Can we do that?

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof:

What does it mean for f to be injective?

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Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

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Let $x_1 = -1$ and $x_2 = +1$.

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so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required.

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

	To ***prove*** that this is true…	
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	
$A \to B$	Assume A is true, then prove B is true.	
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$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	
$\neg A$	Simplify the negation, then consult this table on the result.	

Proof 2: Proving a function is surjective.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

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What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}.\ \exists x \in \mathbb{R}.\ f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

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Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y / 2$.

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Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y / 2$. Then we see that

$$f(x) = f(y / 2)$$

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Proof 3: Proving a function is not surjective.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

What does it mean for g to be surjective?

$$\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$

What is the negation of the above statement?

$$\neg \forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$
$$\exists n \in \mathbb{N}.\ \neg \exists m \in \mathbb{N}.\ g(m) = n$$
$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n$$

Therefore, we need to find a natural number n where, regardless of which m we pick, we have $g(m) \neq n$.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

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Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$.

Our overall goal is to prove

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n.$$

We just made our choice of n. Therefore, we need to prove

$$\forall m \in \mathbb{N}.\ g(m) \neq n.$$

We'll therefore pick an arbitrary $m \in \mathbb{N}$, then prove that $g(m) \neq n$.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$.

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Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$. We need to show that $g(m) \neq n$.

Notice that $g(m) = 2m$ is even, while 137 is odd.

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Notice that $g(m) = 2m$ is even, while 137 is odd. Therefore, we have $g(m) \neq 137$, as required.

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